MAT 4575 Devoir #5 M. ALVO

Aut 11

1\. 8.2.9 Soit $Y = \Sigma X_i$. Quand $EX = \theta$, $Y \simeq N(n\theta, n\theta(1-\theta))$.

(3 points) La fonction de puissance pour tester $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1 > \theta_0$,

$\gamma(\theta) = P_\theta(Y \geq c)$.

$$\gamma(\theta_0) = P\left(\frac{Y - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} \geq \frac{c - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} \right) = \alpha$$

$$\Rightarrow \quad \frac{c - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = z_\alpha \qquad (1)$$

Aussi

$$\gamma(\theta_1) = P\left(\frac{Y - n\theta_1}{\sqrt{n\theta_1(1-\theta_1)}} \geq \frac{c - n\theta_1}{\sqrt{n\theta_1(1-\theta_1)}} \right) = \beta$$

$$\Rightarrow \quad \frac{c - n\theta_1}{\sqrt{n\theta_1(1-\theta_1)}} = z_\beta \qquad (2)$$

$$(1) \quad c - n\theta_0 = \sqrt{n}\sqrt{\theta_0(1-\theta_0)}\, z_\alpha$$

$$(2) \quad c - n\theta_1 = \sqrt{n}\sqrt{\theta_1(1-\theta_1)}\, z_\beta$$

$$(1) - (2) \quad n(\theta_1 - \theta_0) = \sqrt{n}\left[z_\alpha \sqrt{\theta_0(1-\theta_0)} - z_\beta \sqrt{\theta_1(1-\theta_1)} \right]$$

$$\Longrightarrow \quad n = \frac{\left[z_\alpha \sqrt{\theta_0(1-\theta_0)} - z_\beta \sqrt{\theta_1(1-\theta_1)} \right]^2}{(\theta_1 - \theta_0)^2}$$

Pour $z_\alpha = 1.645$, $z_\beta = -1.282$, $\theta_0 = \frac{1}{20}$, $\theta_1 = \frac{1}{10}$,

$$n = 220.89 \simeq 221$$

2\. 8.2.11 C'est un membre de la famille exponentielle avec $p(\theta) = \theta$, $K(x) = \ln x$, $p(\theta) = \ln\theta$

Cette famille possède un mlr en $\Sigma K(x_i) = \Sigma \ln x_i = \ln(\Pi x_i)$

(2 points) Donc le UMP rejette quand $\Pi X_i \leq c$ puisque $\theta > 0$.

2. # 8.3.10

(3 points)

$$\Lambda = \frac{\max_{\theta} \frac{1}{\theta^n} \cdot I(\max x_i < \theta)}{\max_{H_1} \frac{1}{\theta^n} e^{-\sum x_i/\theta}} = \frac{(1/\hat{\theta}^n)}{\frac{1}{\bar{x}^n} e^{-n}} \qquad \hat{\theta} = \max x_i$$

$$\Rightarrow \text{Rej. if} \quad = \left(\frac{\hat{\theta}}{\bar{x}}\right)^n e^n \le k$$

$$\text{i.e.} \qquad \frac{\hat{\theta}}{\bar{x}} < k$$

3. 8.3.12

$$H_0: \mu_1 = \mu_2 = 0$$

(3 points)

$$H_1: \mu_1 \neq 0, \ \mu_2 \neq 0$$

$$\max_{\mu_1 = \mu_2 = 0} L(\mu_1, \mu_2) = \left(\frac{1}{2\pi}\right)^n \left(\frac{1}{\hat{\sigma}^2}\right)^n \exp - \frac{\left(\sum^n x_i^2 + \sum^n y_i^2\right)}{2\hat{\sigma}^2}$$

$$\text{où} \qquad \hat{\sigma}^2 = \frac{\sum x_i^2 + \sum y_i^2}{2n}$$

$$\max_{\mu_1, \mu_2} L(\mu_1, \mu_2) = \left(\frac{1}{2\pi}\right)^n \left(\frac{1}{\hat{\hat{\sigma}}^2}\right)^n \exp - \frac{\sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y})^2}{2\hat{\hat{\sigma}}^2}$$

$$\text{où} \qquad \hat{\hat{\sigma}}^2 = \frac{\sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y})^2}{2n}$$

$$\therefore \Lambda = \left(\frac{1}{\hat{\sigma}^2}\right)^n \left(\frac{\hat{\hat{\sigma}}^2}{1}\right)^n \le k$$

$$\Rightarrow \qquad \frac{\sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y})^2}{\sum x_i^2 + \sum y_i^2} \le k$$

$$\frac{\sum(x_i-\bar{x})^2+\sum(y_i-\bar{y})^2}{\sum(x_i-\bar{x})^2+\sum(y_i-\bar{y})^2+n(\bar{x}^2+\bar{y}^2)} \leq k$$

Rej. when

$$\frac{n(\bar{x}^2+\bar{y}^2)}{\sum(x_i-\bar{x})^2+\sum(y_i-\bar{y})^2} > k$$

On sait que $\frac{\sqrt{n}\,\bar{X}}{\sigma} \sim N(0,1)$, $\frac{\sqrt{n}\,\bar{Y}}{\sigma} \sim N(0,1)$

$$\Rightarrow \quad \frac{n\bar{X}^2+n\bar{Y}^2}{\sigma^2} \sim \chi^2_2$$

Aussi

$$\frac{\sum(X_i-\bar{X})^2+\sum(Y_i-\bar{Y})^2}{\sigma^2} \sim \chi^2_{2(n-1)} \quad \text{indep.}$$

$$\Rightarrow \quad F = \frac{\frac{n(\bar{X}^2+\bar{Y}^2)}{2\sigma^2}}{S^2/\sigma^2} \sim F_{2,\,2(n-1)} \quad \text{sous } H_0.$$

c) Quand H_1 est vraie, $\frac{\sqrt{n}}{\sigma}(\bar{X}-\mu_1) \sim N(0,1)$

$$\frac{\sqrt{n}}{\sigma}(\bar{Y}-\mu_2) \sim N(0,1)$$

$$\Rightarrow \quad \frac{n}{\sigma^2}\left[(\bar{X}-\mu_1)^2+(\bar{Y}-\mu_2)^2\right] \sim \chi^2_2$$